PURDUE UNIVERSITY

DEPARTMENT OF STATISTICS

DIVISION OF MATHEMATICAL SCIENCES

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ON A LOCALLY OPTIMAL PROCEDURE BASED ON RANKS
FOR COMPARISON OF TREATMENTS WITH A CONTROL*

by

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1. Indroduction. Let $\pi_1,\ldots,\pi_k$ be independent populations representing k experimental treatments and let π_0 be the control treatment. Let $f(x,\theta_i)$ denote the density of π_i, $i=0,1,\ldots,k$. Any population π_i is said to be superior to the control if $\theta_i > \theta_0$, and inferior otherwise. While θ_0 is not known, we have, based on past experience, a fair idea of it so as to assume that $\theta_0 \leq \theta_0^*$, a known quantity. Following the earlier setup of Gupta, Huang and Nagel [1] and Huang and Panchapakesan [3], who have studied locally optimal rules based on ranks for selecting the best population, we assume that the functional form of $f(x,\theta)$ is known but for the value of the parameter. We seek a procedure based on ranks in view of the usual considerations of robustness against possible deviations from the model. We are interested in selecting a subset (possibly empty) of the k experimental treatments consisting of those that are superior to the control.

Let X_{ij}, $j=1,\ldots,n$, be independent observations from π_i, $i=0,1,\ldots,k$. Let R_{ij} denote the rank of X_{ij} in the pooled sample of $N = (k+1)n$ observations. The smallest observation has rank 1 and the largest rank N. Let $x_1 \leq x_2 \leq \cdots \leq x_N$ denote the ordered observations. A rank configuration is an N-tuple $\Delta = (\Delta_1,\ldots,\Delta_N)$, $\Delta_i \in \{1,2,\ldots,k\}$, where $\Delta_i = j$ means that the

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ith smallest observation in the pooled sample comes from π_j. Let $\mathcal{C} = \{\Delta\}$ denote the set of all rank configurations for fixed k and n. For fixed Δ, let $\mathcal{X}_\Delta = \{\underline{x} \in \mathcal{X} | \Delta_{\underline{x}} = \Delta\}$, where $\mathcal{X} = \{\underline{x}: \underline{x} = (x_1,\ldots,x_N)\}$ and $\Delta_{\underline{x}}$ denotes the rank configuration of $\underline{x} = (x_1,\ldots,x_N)$. A decision rule δ based on the observed rank configuration Δ is a k-tuple defined by $\delta \equiv \delta(\Delta) = \{\delta_1(\Delta),\ldots,\delta_k(\Delta)\}$, where $\delta_i(\Delta)$ is the (conditional) probability that π_i is selected as a superior population.

Let $\underline{\theta} = (\theta_0,\theta_1,\ldots,\theta_k)$ and $\Omega = \{\underline{\theta} | \theta_0 \le \theta^*\}$. Define

$$\Omega_0 = \{\underline{\theta} | \theta_i = \theta_0 \le \theta_0^*, i=1,\ldots,k\} \text{ and } \Omega_{i0}^* = \{\underline{\theta} | \theta_j = \theta_0^* < \theta_i, j\neq i\},\ i=1,\ldots,k.$$

We are interested in the class of rules δ satisfying

$$P_{\underline{\theta}}\{\pi_i \text{ is selected} | \underline{\theta} \in \Omega_0\} \le \gamma \text{ for } i=1,\ldots,k. \tag{1.1}$$

In this class, we seek a locally optimal rule in the sense that it maximizes

$$\sum_{i=1}^{k} \frac{\partial}{\partial\theta_i} P_{\underline{\theta}}\{\pi_i \text{ is selected} | \underline{\theta} \in \Omega_{i0}^*\}\Big|_{\theta_i=\theta_0^*}. \tag{1.2}$$

Let $P_{\underline{\theta}}(\Delta)$ denote the probability of realizing the rank configuration Δ. Then (1.1) can be written as

$$\sum_{\mathcal{C}} \delta_i(\Delta) P_{\underline{\theta}_0}(\Delta) \le \gamma \text{ for } i=1,\ldots,k, \tag{1.3}$$

where $\underline{\theta}_0 = (\theta_0,\ldots,\theta_0) \in \Omega_0$ and the expression (1.2) is equal to

$$\sum_{i=1}^{k} \frac{\partial}{\partial\theta_i} \sum_{\mathcal{C}} \delta_i(\Delta) P_{\underline{\theta}^{(i)}}(\Delta)\Big|_{\theta_i=\theta_0^*} \quad \text{where } \underline{\theta}^{(i)} \text{ denotes} \tag{1.4}$$

a point in Ω_{i0}^*. The condition (1.3) corresponds to controlling error probabilities and the optimality condition in (1.4) reflects the sensitivity of the rule when all but one population are not distinctly superior $(\theta_j=\theta_0^*, j\neq i)$ and the remaining one is in a neighborhood of the others but distinctly superior $(\theta_i > \theta_0^*)$.

2. Derivation of a locally optimal rule. We assume that the density $f(x,\theta)$ satisfies the following set of regularity conditions: (i) $f(x,\theta)$ is absolutely continuous in θ for almost every x, (ii) $f(x,\theta)$ is continuously differentiable with respect to θ for almost every x, and (iii) $\dot{f}(x,\theta) = \frac{\partial}{\partial\theta} f(x,\theta)$ is integrable.

Now, the probability $P_{\underline{\theta}}(\Delta)$ of realizing the rank configuration Δ under $\underline{\theta} \in \Omega$ is by

$$(2.1) \qquad P_{\underline{\theta}}(\Delta) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \prod_{i=1}^{N} f(x_i, \theta_{\Delta_i}) dx_1 \ldots dx_N .$$

We note that $P_{\underline{\theta}_0}(\Delta)$ is independent of the common value θ_0 of the parameters and is equal to $1/N!$. Thus, the condition (1.3) becomes

$$(2.2) \qquad \frac{1}{N!} \sum_{C} \delta_i(\Delta) \leq \gamma \text{ for } i=1,\ldots,k.$$

For $\underline{\theta}^{(i)} \in \Omega^*_{i0}$, it can be easily seen that

$$(2.3) \qquad \frac{\partial}{\partial\theta_i} P_{\underline{\theta}^{(i)}}(\Delta)\Big|_{\theta_i=\theta_0^*}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \left\{ \prod_{e=1}^{N} f(x_e, \theta_0^*) \right\} \sum_{\substack{j \\ \Delta_j = i}} \frac{\dot{f}(x_j, \theta_0^*)}{f(x_j, \theta_0^*)} dx_1 \ldots dx_N$$

$$= \sum_{\substack{j \\ \Delta_j = i}} \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \dot{f}(x_j, \theta_0^*) \prod_{\substack{i=1 \\ i \neq j}}^{N} f(x_i, \theta_0^*) dx_1 \ldots dx_N$$

$$= A_i(\Delta, \theta_0^*), \text{ say.}$$

Thus we wait to derive a rule δ which satisfies (2.2) and which, among all rules that satisfy (2.2), maximizes

$$(2.4) \qquad \sum_{i=1}^{k} \sum_{C} \delta_i(\Delta) A_i(\Delta, \theta_0^*).$$

The following theorem provides such a rule.

Theorem 2.1. Under all the assumptions stated previously, a rule $\delta^0(\Delta)$ which satisfies (1.1) [or equivalently (2.2)] and which, among all rules satisfying (1.1), maximizes (1.2) [or equivalently (2.4)] is given by

$$(2.5) \qquad \delta_i^0(\Delta) = \begin{cases} 1 & > \\ \rho & A_i(\Delta, \theta_0^*) = c_i/N! \\ 0 & < \end{cases}$$

where $0 < \rho < 1$ and c_i are determined such that

$$(2.6) \qquad \frac{1}{N!} \sum_{C} \delta_i^0(\Delta) = \gamma.$$

Proof. Let $\delta(\Delta)$ be any rule other than $\delta^0(\Delta)$ satisfying (2.2). Then

$$\sum_{i=1}^{k} \sum_{C} \left\{\delta_i(\Delta) - \delta_i^0(\Delta)\right\}\left\{A_i(\Delta, \theta_0^*) - \frac{c_i}{N!}\right\} \leq 0.$$

Now, using (2.2) and (2.6), we get

$$\sum_{i=1}^{k} \sum_{C} \delta_i(\Delta) A_i(\Delta, \theta_0^*) \leq \sum_{i=1}^{k} \sum_{C} \delta_i^0(\Delta) A_i(\Delta, \theta_0^*).$$

This proves the theorem.

We note that this locally optimal rule is based on weighted rank sums using the scores

$$(2.7) \qquad B_i = \frac{N!}{(i-1)!(N-i)!} \int_0^1 u^{i-1}(1-u)^{N-i} \phi(u, f, \theta_0^*) du,$$

where

$$(2.8) \qquad \phi(u, f, \theta_0^*) = \frac{\dot{f}(F^{-1}(u, \theta_0^*), \theta_0^*)}{f(F^{-1}(u, \theta_0^*), \theta_0^*)}$$

which in general depends on θ_0^*. However, it is independent of θ_0^* if it is a location or scale parameter.

3. A special case. One can specialize the rule δ_0 given by (2.5) to specific densities $f(x,\theta)$. An important special case arises when $f(x,\theta)$ is the logistic density $f(x,\theta) = e^{-(x-\theta)}/[1+e^{-(x-\theta)}]^2$, $-\infty < x < \infty$, $-\infty < \theta < \infty$. In this case, $\phi(u,f,\theta) = 2u - 1$ which leads to equally spaced scores and B_i is of the form $B_i = a + ib$, where $b > 0$. Consequently, the rule δ_0 is given by

$$(3.1) \qquad \delta_i^0(\Delta) = \begin{cases} 1 & \\ \rho & \sum_{j=1}^{n} R_{ij} \gtreqless c/N! \\ 0 & \end{cases}$$

where $0 < \rho < 1$ and c are determined by

$$(3.2) \qquad P_{\underline{\theta}_0^*}\left\{\sum_{j=1}^{n} R_{ij} > c/N!\right\} + \rho P_{\underline{\theta}_0^*}\left\{\sum_{j=1}^{n} R_{ij} = c/N!\right\} = \gamma .$$

The values of ρ and c can be obtained from tables for Wilcoxon two-sample rank-sum statistic.

4. Some remarks. Nagel [4] defined just rules for selecting the best population. This concept can be applied also to the problem of selecting populations that are better than control. In our setup, it means that the probability of selecting π_i is nondecreasing if all the observations from π_i are increased and the observations from all other populations are decreased. The rule δ^0 defined by (2.5) is just if B_i is nondecreasing in i. In the case of location parameters, this monotonicity of B_i is equivalent to saying that $f(x)$ is strongly unimodel, i.e., $-\log f(x)$ is convex (see [2], p.20). In the special case of logistic densities, the rule δ^0 given by (3.1) is just.

Though θ_0 is not known, we have assumed that an upper bound θ_0^* is known. If θ_0 is known, then in stating the optimality requirement, θ_0^* is replaced by θ_0.

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Let $\pi_1,\ldots,\pi_k$ be independent populations representing k experimental treatments and let π_0 be the control treatment. Let $f(x,\theta_i)$ denote the density of π_i, i=0,1, ...,k, satisfying certain regularity conditions. Any population π_i is said to be superior to the control if $\theta_i > \theta_0$, and inferior otherwise. We assume that the

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value of θ_0 is unknown but a good upper bound θ_0^* is known. We are interested in selecting a subset (possibly empty) of the k experimental treatments consisting of the ones that are superior to the control. Though the functional form of $f(x,\theta)$ is assumed to be known, we derive a procedure based on ranks in view of the usual considerations of robustness against possible deviations from the model. In deriving the rule, we control the error probabilities and maximize the ability to detect a superior population in a certain local sense. The rule is based on the ranks of the observations in a pooled sample of $(k+1)n$ [n from each population] observations. In the special case of logistic density, this rule is expressed in terms of the rank-sum statistics.